Yang-Mills Replacement

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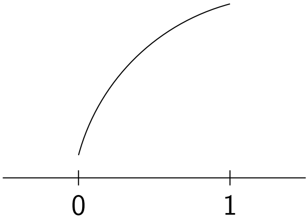
Schwarz Alternating Method

Example

Let $f : [0, 1] \to \mathbb{R}$. We want to make f harmonic while fixing its boundary values.



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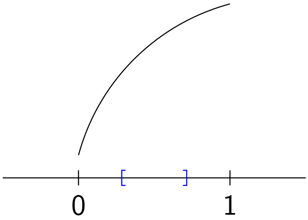

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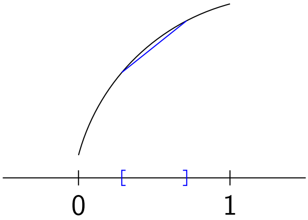

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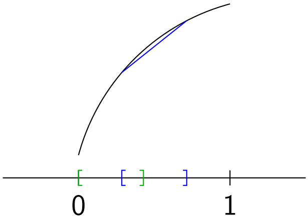

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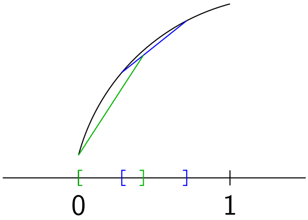

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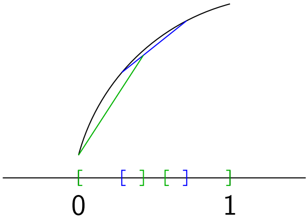

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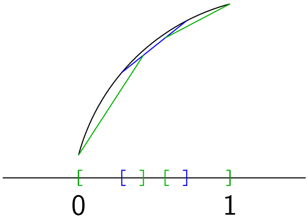

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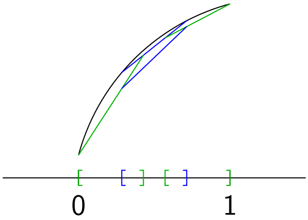

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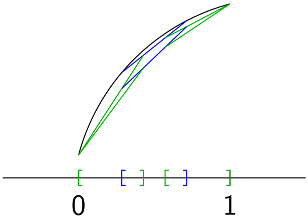

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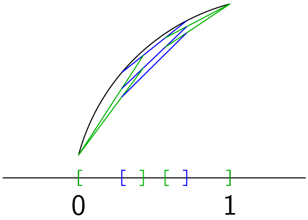

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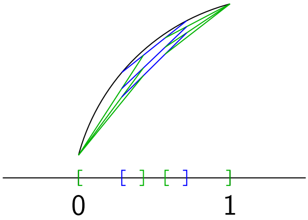

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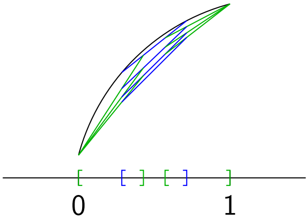

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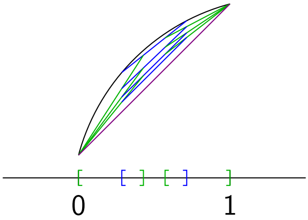

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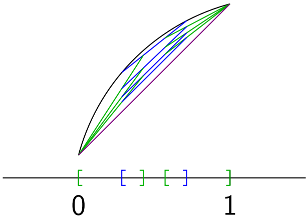

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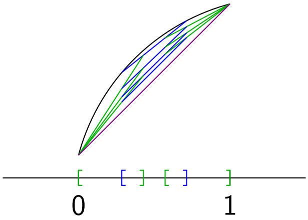

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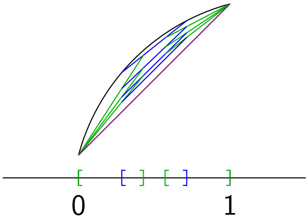

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- I showed that one can similarly locally replace connections on 4-manifolds with Yang-Mills connections, with bounds.

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- Yang-Mills replacement has parallels with Yang-Mills gradient flow.
  - Ability to choose balls gives more control.

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$(\Delta u)^\top = (d^* du)^\top = 0.$	$d_A^* F_A = 0.$

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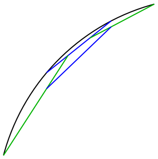

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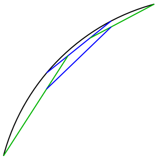

- Our boundary values are $L^2_{1/2}(\partial B^4)$, and solutions are $L^2_1(B^4)$.

Local Yang-Mills Replacement

Theorem (YBK)

- *For any $L_1^2(B^4)$ low-energy connection A, there exists a low-energy $L_1^2(B^4)$ Yang-Mills connection B, unique up to gauge, such that $i^*A = i^*B$.*

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- *The linear interpolation between A and B has monotone decreasing energy.*
  - *Equality if and only if A is already Yang-Mills.*

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$\phi \mapsto (d^* d\phi, i^*\phi)$ is invertible.	$\alpha \mapsto (d^* d\alpha, i^*\alpha)$ is not invertible.
	Solution: Can get $d^*\alpha = 0$ by choosing a good gauge.

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- *$\tilde{A}$ is in Dirichlet Coulomb gauge, that is,*
  - *$d^* \tilde{a} = 0$ on B^4, and*
  - *$d^*_{\partial B^4} i^* \tilde{a} = 0$ on ∂B^4.*
- $\|\tilde{a}\|_{L^2_1(B^4)} \leq C \|F_A\|_{L^2(B^4)}$.

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Theorem (Dirichlet Uhlenbeck gauge fixing, YBK)

Any low-energy $L^2_1(B^4)$ connection A is gauge equivalent to an $L^2_1(B^4)$ connection $\tilde{A} = d + \tilde{a}$ such that:

- *$\tilde{A}$ is in Dirichlet Coulomb gauge, that is,*
  - *$d^*\tilde{a} = 0$ on B^4, and*
  - *$d^*_{\partial B^4} i^*\tilde{a} = 0$ on ∂B^4.*
- $\|\tilde{a}\|_{L^2_1(B^4)} \leq C \|F_A\|_{L^2(B^4)}$.

The boundary condition $d^*_{\partial B^4} i^*\tilde{a} = 0$ is preserved under gauge transformations satisfying Dirichlet boundary conditions.

Solving the Dirichlet Problem

We want to invert the map $A \mapsto (d_A^* F_A, i^* A)$ near the trivial connection, where A is an $L_1^2(B^4)$ connection.

- The linearization is $\alpha \mapsto (d^* d\alpha, i^*\alpha)$, which is not invertible.

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  - The linearization is now equal to $\alpha \mapsto (\Delta\alpha, i^*\alpha)$.

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  - $\alpha \mapsto (\Delta\alpha, i^*\alpha, i^* d^*\alpha)$ is invertible, but only for $\alpha \in L_2^2(B^4)$.

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- Restricting to $\ker d^*$ gives an isomorphism

$$(d^* d, i^*) \colon L_2^2(B^4) \cap \ker d^* \to L^2(B^4) \cap \operatorname{range}(d^*) \times L_{3/2}^2(\partial B^4).$$

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- But in the regularity we want,

$$(d^* d, i^*)\colon L_1^2(B^4) \cap \ker d^* \to L_{-1}^2(B^4) \cap \mathrm{range}(d^*) \times L_{1/2}^2(\partial B^4)$$

is not injective.

Solving the Dirichlet Problem

We want to invert the map $A \mapsto (d_A^* F_A, i^* A)$ near the trivial connection, where A is an $L_1^2(B^4)$ connection.

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  is not injective.
- Solution: Use a target space slightly larger than $L_{-1}^2(B^4)$.

Solving the Dirichlet Problem

Choosing the Target Banach Space

We want to invert the map $A \mapsto (d_A^* F_A, i^* A)$ near the trivial connection, where A is an $L_1^2(B^4)$ connection.

- The linearization is $\alpha \mapsto (d^* d\alpha, i^*\alpha)$.

Solving the Dirichlet Problem

Choosing the Target Banach Space

We want to invert the map $A \mapsto (d_A^* F_A, i^* A)$ near the trivial connection, where A is an $L_1^2(B^4)$ connection.

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Definition

Dual

$$L_1^2(B^4)^0 = \{\alpha \in L_1^2(B^4) \mid \alpha|_{\partial B^4} = 0\} \qquad L_{-1}^2(B^4)$$

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L_1^2(B^4)^0 = \{\alpha \in L_1^2(B^4) \mid \alpha|_{\partial B^4} = 0\} \arrow[d, hook] & L_{-1}^2(B^4) \\
L_1^2(B^4)^{\mathrm{rel}} = \{\alpha \in L_1^2(B^4) \mid i^*\alpha = 0\} &
\end{tikzcd}
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\end{array}
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Solving the Dirichlet Problem

Choosing the Target Banach Space

We want to invert the map $A \mapsto (d_A^* F_A, i^* A)$ near the trivial connection, where A is an $L^2_1(B^4)$ connection.

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\end{tikzcd}
$$

- d^*d is bounded as an operator $d^*d \colon L^2_1(B^4) \to L^2_{-1}(B^4)$.

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- $d^* d$ is bounded as an operator $d^* d \colon L_1^2(B^4) \to L_{-1}^2(B^4)$.
- $d^* d$ is still bounded as $d^* d \colon L_1^2(B^4) \to L_{-1}^2(B^4)^{\mathrm{rel}}$.

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- $\alpha \mapsto (d^*d\alpha, i^*\alpha)$ is invertible as an operator

$$(d^*d, i^*) \colon L_1^2(B^4) \cap \ker d^* \to L_{-1}^2(B^4)^{\mathrm{rel}} \cap \mathrm{range}(d^*) \times L_{1/2}^2(\partial B^4).$$

Solving the Dirichlet Problem

Projecting to range(d^*)

We want to invert the map $A \mapsto (d_A^* F_A, i^* A)$ near the trivial connection, where A is an $L_1^2(B^4)$ connection.

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- Problem: $d_A^* F_A$ does not lie in range(d^*) in general.

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  - Let π_{d^*} be the $L^2(B^4)$-projection to range(d^*).
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  - The linearization of $A \mapsto (\pi_{d^*} d_A^* F_A, i^* A)$ at the trivial connection is $(\pi_{d^*} d^* d\alpha, i^*\alpha) = (d^* d\alpha, i^*\alpha)$.

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- Given A_∂ small in the $L_{1/2}^2(\partial B^4)$ norm, we can solve

$$\begin{aligned} \pi_{d^*} d_A^* F_A &= 0 \quad \text{on } B^4 \\ i^* A &= A_\partial \quad \text{on } \partial B^4 \end{aligned}$$

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$$\begin{aligned} \pi_{d^*} d_A^* F_A &= 0 \quad \text{on } B^4 \\ i^* A &= A_\partial \quad \text{on } \partial B^4 \end{aligned}$$

- We also have $d^* a = 0$ and that a is small in $L^2_1(B^4)$.

Solving the Dirichlet Problem

Concluding that the connection minimizes energy

- We have found a $B = d + b$ such that $\pi_{d^*} d_B^* F_B = 0$ and b is small in $L_1^2(B^4)$.

Solving the Dirichlet Problem

Concluding that the connection minimizes energy

- We have found a $B = d + b$ such that $\pi_{d^*} d_B^* F_B = 0$ and b is small in $L_1^2(B^4)$.
- We want to conclude that $d_B^* F_B = 0$.

Solving the Dirichlet Problem

Concluding that the connection minimizes energy

- We have found a $B = d + b$ such that $\pi_{d^*} d_B^* F_B = 0$ and b is small in $L_1^2(B^4)$.
- We want to conclude that $d_B^* F_B = 0$.
- In higher regularity $b \in L_2^2(B^4)$, given $\pi_{d^*} d_B^* F_B = 0$, we can prove an inequality of the form

$$\left\| d_B^* F_B \right\|_{L^2(B^4)} \leq C \left\| b \right\|_{L^4(B^4)} \left\| d_B^* F_B \right\|_{L^2(B^4)} .$$

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  - Conclude that $d_B^* F_B = 0$ as long as $\|b\|_{L^4(B^4)}$ is small.

Solving the Dirichlet Problem

Concluding that the connection minimizes energy

- We have found a $B = d + b$ such that $\pi_{d^*} d_B^* F_B = 0$ and b is small in $L_1^2(B^4)$.
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  - Conclude that $d_B^* F_B = 0$ as long as $\|b\|_{L^4(B^4)}$ is small.
- This argument fails at $b \in L_1^2(B^4)$ regularity.

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- In higher regularity $b \in L_2^2(B^4)$, given $\pi_{d^*} d_B^* F_B = 0$, we can prove an inequality of the form

$$\left\| d_B^* F_B \right\|_{L^2(B^4)} \leq C \left\| b \right\|_{L^4(B^4)} \left\| d_B^* F_B \right\|_{L^2(B^4)}.$$

  - Conclude that $d_B^* F_B = 0$ as long as $\|b\|_{L^4(B^4)}$ is small.
- This argument fails at $b \in L_1^2(B^4)$ regularity.
- Instead, we directly show that B locally minimizes energy and is thus Yang-Mills, using the inequality

$$\|A - B\|_{L_1^2(B^4)}^2 \leq C \left(\|F_A\|_{L^2(B^4)}^2 - \|F_B\|_{L^2(B^4)}^2 \right).$$

Solving the Dirichlet Problem

Concluding that the connection minimizes energy

- We have found a $B = d + b$ such that $\pi_{d^*} d_B^* F_B = 0$ and b is small in $L_1^2(B^4)$.
- We want to conclude that $d_B^* F_B = 0$.
- In higher regularity $b \in L_2^2(B^4)$, given $\pi_{d^*} d_B^* F_B = 0$, we can prove an inequality of the form

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- This argument fails at $b \in L_1^2(B^4)$ regularity.
- Instead, we directly show that B locally minimizes energy and is thus Yang-Mills, using the inequality

$$\|A - B\|^2_{L_1^2(B^4)} \leq C \left(\|F_A\|^2_{L^2(B^4)} - \|F_B\|^2_{L^2(B^4)} \right).$$

  - The inequality holds even if B only satisfies $\pi_{d^*} d_B^* F_B = 0$, along with assumptions of small energy, matching on the boundary, and Dirichlet Coulomb gauge.

Towards Global Yang-Mills Replacement

We want to repeat Yang-Mills replacement on balls covering the manifold X to obtain a global Yang-Mills connection in the limit.

Bubbling

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- Yang-Mills replacement requires small energy on each ball.
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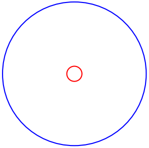

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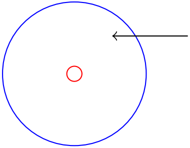

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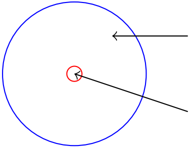
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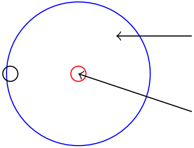

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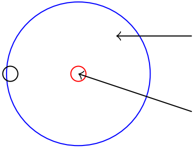

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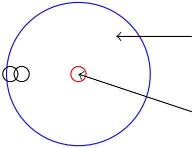

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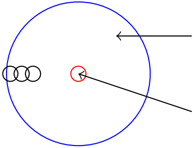

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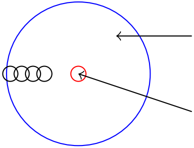

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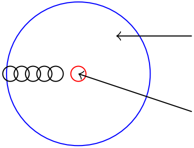

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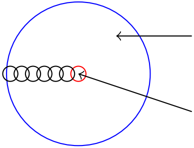

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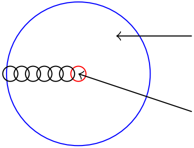

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- Potential solution: Moving energy costs energy.

$$\left\| \tilde{A} - \tilde{B} \right\|^2_{L^2_1(B^4)} \leq C \left(\|F_A\|^2_{L^2(B^4)} - \|F_B\|^2_{L^2(B^4)} \right).$$

Towards Global Yang-Mills Replacement

We want to repeat Yang-Mills replacement on balls covering the manifold X to obtain a global Yang-Mills connection in the limit.

Limit cycles in the space of connections

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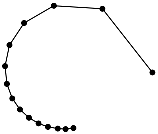

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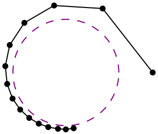

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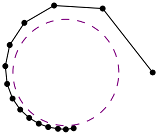

- Differences must go to zero by

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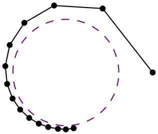

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  - Not strong enough to guarantee convergence.

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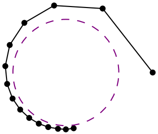

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- Can still use weak subsequence convergence.

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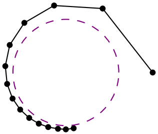

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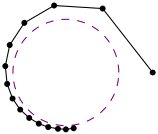

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- Łojasiewicz inequality.

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- Given a compact family of connections, we can choose the sequence of balls uniformly for the entire family.

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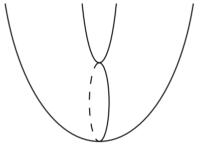
Yang-Mills connections with positive Morse index



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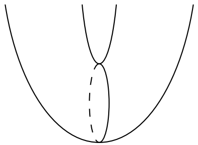

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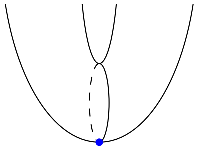

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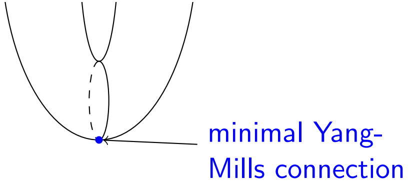

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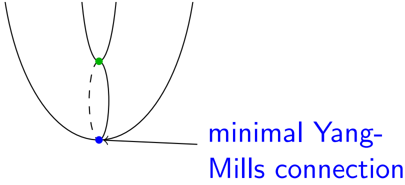

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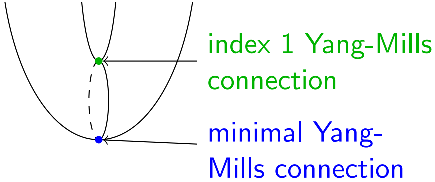

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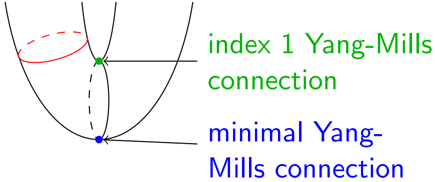

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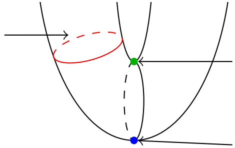
Yang-Mills connections with positive Morse index

Compact family of connections



index 1 Yang-Mills connection

minimal Yang-Mills connection

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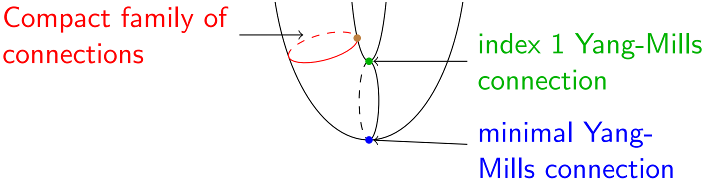

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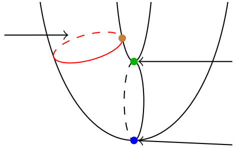

- Global Yang-Mills replacement cannot be continuous in the initial data.

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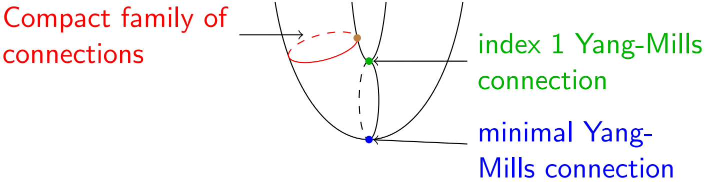

- Global Yang-Mills replacement cannot be continuous in the initial data.
  - Might be continuous if the initial data is below all non-minimal critical points.

Thank You

Acknowledgments

- Tom Mrowka
- National Science Foundation
- Department of Defense, NDSEG

Selected References

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Towards Global Yang-Mills Replacement

Bonus Slide

We want to repeat Yang-Mills replacement on balls covering the manifold X to obtain a global Yang-Mills connection in the limit.

Discontinuous normal components

- Only the tangential components of the replacement match the original connection on ∂B^4.
- The normal derivative of the normal component of the new connection is not $L^2(X)$ across ∂B^4.
- After local Yang-Mills replacement, the global connection is no longer $L^2_1(X)$.
- Solution: With a different choice of gauge on a slightly larger ball, the connection becomes $L^2_1(X)$.