Uncovering the Lagrangian of a system from discrete observations

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Background

The Problem

Given discrete measurements of a Lagrangian system, can we recover the Lagrangian?

The Discrete Euler-Lagrange Equations

Given a system with Lagrangian $L(x, v)$, we can discretize the action with time step τ by summing the discrete Lagrangian

$$L_d(x, y) = \tau \cdot L\left(\frac{x+y}{2}, \frac{y-x}{\tau}\right).$$

The principle of stationary action yields the discrete Euler-Lagrange equations

$$D_2 L_d(x, y) + D_1 L_d(y, z) = 0,$$

relating any three consecutive points x, y, and z on a discrete trajectory.

Recovering the Discrete Lagrangian

The Problem

Given a pair of points (x_0, y_0), we would like to use data points on trajectories that pass nearby to estimate the Taylor expansion of the discrete Lagrangian L_d at (x_0, y_0).

A Caveat

Two Lagrangians might be equivalent in the sense that they yield the same equations of motion.

- For example, if $L(x, y)$ is a discrete Lagrangian, then the Lagrangian

$$L'(x, y) = \alpha L(x, y) + \beta(y^2 - x^2) + \gamma(y - x) + \delta$$

  produces the same discrete Euler-Lagrange equations, for any choice of α, β, γ, and δ.
- Trajectory data can't distinguish between equivalent Lagrangians, nor would it be useful to do so.

A Second-Order Approximation

Taylor Expansion of the Discrete Lagrangian

Given a pair of points (x_0, y_0), we approximate L_d with its second-degree Taylor polynomial at (x_0, y_0). We rewrite it in the form

$$L_d \approx a(x-p)^2 + 2b(x-p)(y-p) + c(y-p)^2 + d_p(x-p) + e_p(y-p) + f_p,$$

where $p = (x_0 + y_0)/2$. The expression in higher dimensions is analogous.

Discrete Euler-Lagrange Equations

For three consecutive points x, y, and z on a trajectory, we can apply the discrete Euler-Lagrange equations $D_2 L_d(x, y) + D_1 L_d(y, z) = 0$ to find

$$0 \approx 2(a+c)(y-p) + 2b(x-p+z-p) + (d_p + e_p).$$

We will use nearby triplets (x, y, z) from our trajectory measurements to estimate $a + c$, b, and $d_p + e_p$.

New Parameters for the Lagrangian

Scaling the Parameters

- In order to estimate the parameters using many data points, we need to assign weights to the parameters appropriately.
- The parameter $d_p + e_p$ has different units from $a + c$ and b.
- Taylor approximations of the discrete Euler-Lagrange equations suggest that an appropriate rescaling of the parameters at (x_0, y_0) is

$$A := (a + c) \left\| y_0 - x_0 \right\|, \quad B := 2b \left\| y_0 - x_0 \right\|, \quad D := d_p + e_p.$$

Discrete Euler-Lagrange Equations

For three consecutive points (x, y, z) on a trajectory, we have

$$0 \approx A(2y - x_0 - y_0) + B(x + z - x_0 - y_0) + D \left\| y_0 - x_0 \right\|.$$

Estimating the Parameters

Discrete Euler-Lagrange Equations

For three consecutive points (x, y, z) on a trajectory, we have

$$0 \approx A(2y - x_0 - y_0) + B(x + z - x_0 - y_0) + D \left\| y_0 - x_0 \right\| .$$

Estimating Lagrangian Parameters with Several Data Points

Given data of consecutive triplets (x_i, y_i, z_i), we estimate A, B, and D up to scaling at a point (x_0, y_0) as follows.

- Construct a matrix M whose rows are

$$w_i \cdot \begin{pmatrix} 2y_i - (x_0 + y_0) & x_i + z_i - (x_0 + y_0) & \left\| y_0 - x_0 \right\| \end{pmatrix} .$$

- For the correct values of the parameters, we will have $M \begin{pmatrix} A \\ B \\ D \end{pmatrix} \approx 0$.
- Estimate A, B, and D by the eigenvector corresponding to the least eigenvalue of $M^T M$.

Assigning Weights to the Data Points

The matrix of coefficients

The ith row of M is

$$w_i \cdot \begin{pmatrix} 2y_i - (x_0 + y_0) & x_i + z_i - (x_0 + y_0) & \|y_0 - x_0\| \end{pmatrix}.$$

Distance

We define the distance bewteen (x_0, y_0) and (x, y) to be

$$\delta((x_0, y_0), (x, y))^2 = \left\| \frac{x+y}{2} - \frac{x_0+y_0}{2} \right\|^2 + \tau_s^2 \left\| \frac{y-x}{\tau} - \frac{y_0-x_0}{\tau} \right\|^2,$$

where τ_s is a parameter and τ is the timestep.

Weights

$$w_i = \exp\left(-\frac{1}{2\sigma^2} \left(\delta((x_0, y_0), (x_i, y_i))^2 + \delta((x_0, y_0), (y_i, z_i))^2 \right) \right),$$

where σ is another parameter.

The Simple Pendulum

The Lagrangian

$$L_d(x,y) = \tau\left(\frac{1}{2}\left(\frac{y-x}{\tau}\right)^2 - \left(1-\cos\left(\frac{x+y}{2}\right)\right)\right).$$

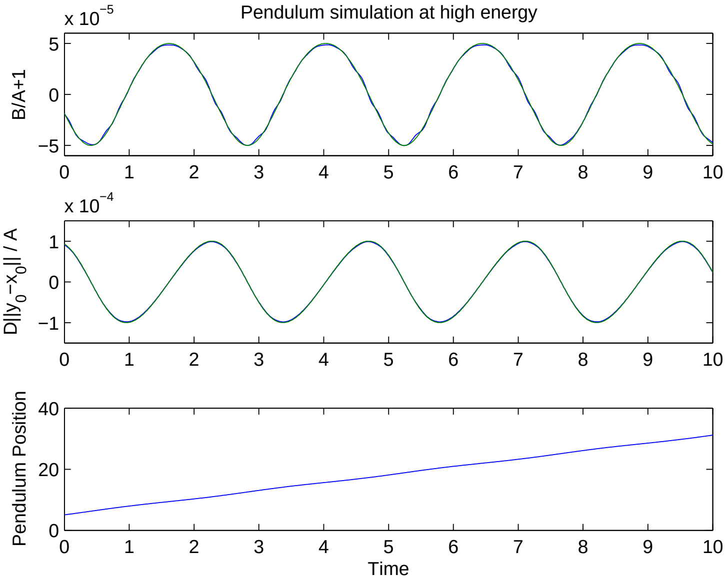
True Values of Lagrangian Parameters

Using a Taylor approximation to the Lagrangian, we find that

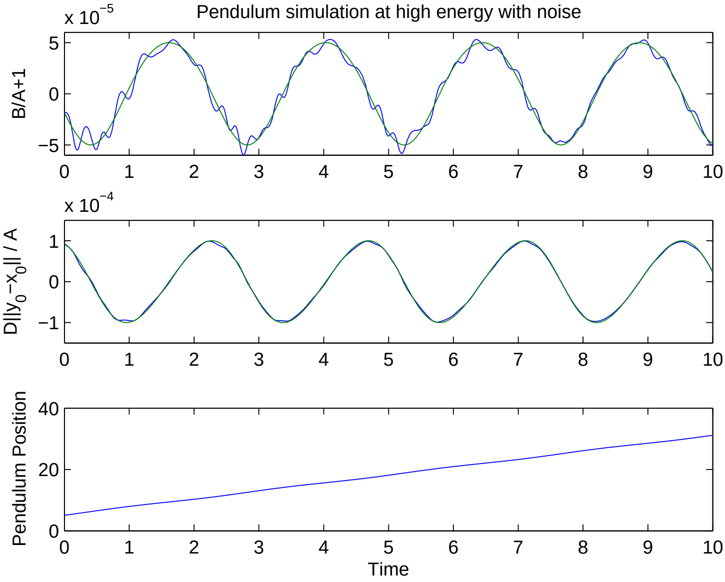
$$\frac{B}{A} = -\frac{4+\tau^2\cos\left(\frac{x_0+y_0}{2}\right)}{4-\tau^2\cos\left(\frac{x_0+y_0}{2}\right)}, \qquad \frac{D}{A} = -\frac{4\tau^2}{\|y_0-x_0\|}\cdot\frac{\sin\left(\frac{x_0+y_0}{2}\right)}{4-\tau^2\cos\left(\frac{x_0+y_0}{2}\right)}.$$

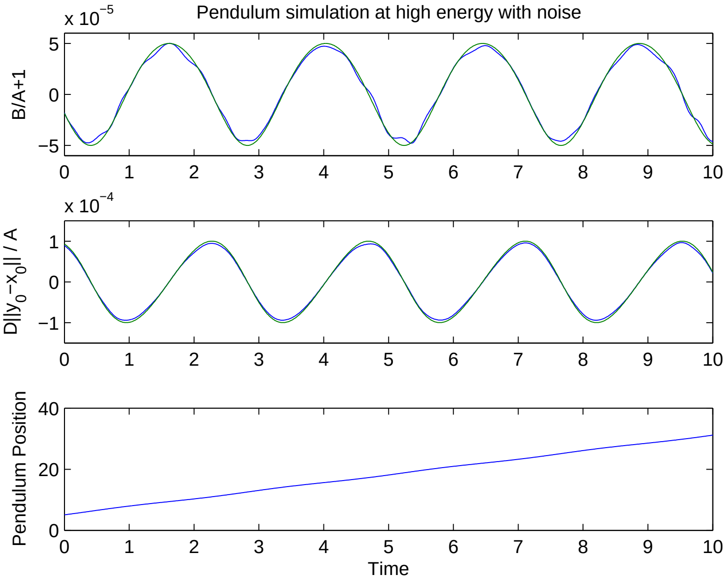
Parameters Computed From Trajectories

I computed the parameters from the trajectories with Matlab. The graphs of $\frac{B}{A}+1$ and $\frac{D}{A}\|y_0-x_0\|$ are on the following slides.



Pendulum simulation at high energy
x 10^−5
B/A+1
5
0
−5
0 1 2 3 4 5 6 7 8 9 10
x 10^−4
D||y0−x0|| / A
1
0
−1
0 1 2 3 4 5 6 7 8 9 10
Pendulum Position
40
20
0
0 1 2 3 4 5 6 7 8 9 10
Time


Pendulum simulation at high energy with noise
x 10^-5
B/A+1
5
0
−5
0 1 2 3 4 5 6 7 8 9 10
x 10^-4
D||y0−x0|| / A
1
0
−1
0 1 2 3 4 5 6 7 8 9 10
Pendulum Position
40
20
0
0 1 2 3 4 5 6 7 8 9 10
Time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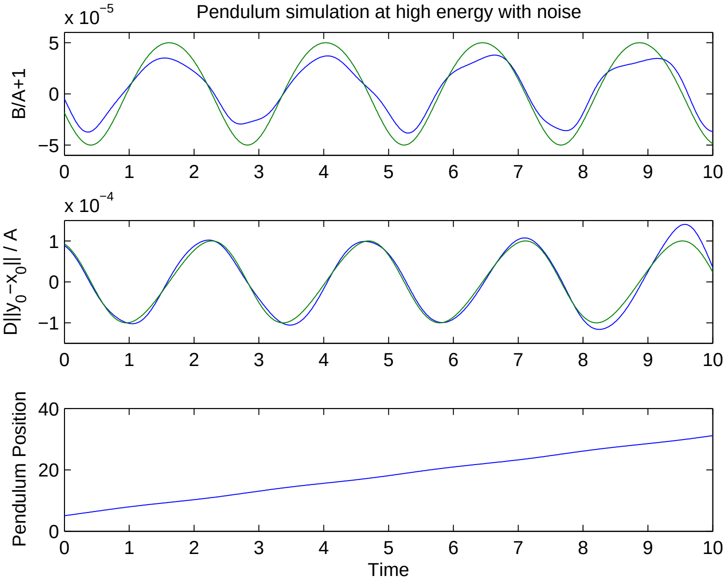
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Time

What Next?

Future Directions

- Recover the Lagrangian from data of several trajectories, and then use it to predict new trajectories.
- Investigate the best choices for τ_s and σ.
- Try adding other kinds of noise to the system.
- Try the method with real data.

Evan S. Gawlik, Patrick Mullen, Dmitry Pavlov, Jerrold E. Marsden, and Mathieu Desbrun, *Geometric, variational discretization of continuum theories*, 2010.

Ari Stern and Mathieu Desbrun, *Discrete geometric mechanics for variational time integrators*, Discrete Differential Geometry: An Applied Introduction, 2006, pp. 75–80.