Numerical Methods in Differential Geometry

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Part 1

Mean curvature flow

Curve shortening flow

$$\frac{d}{dt}\mathbf{x} = -\kappa(\mathbf{x})\mathbf{n}.$$

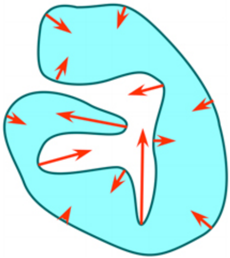

Figure: Curve shortening flow. Image credit: Treibergs. Video credit: Angenent.

Mean curvature flow

$$\frac{d}{dt}\mathbf{x} = -H(\mathbf{x})\mathbf{n}$$

Figure: Mean curvature flow. Video credit: Kovács.

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  - Yes, a torus (Angenent, 1989).
  - Many others (Kapouleas, Kleene, Møller, 2011).

The Angenent torus

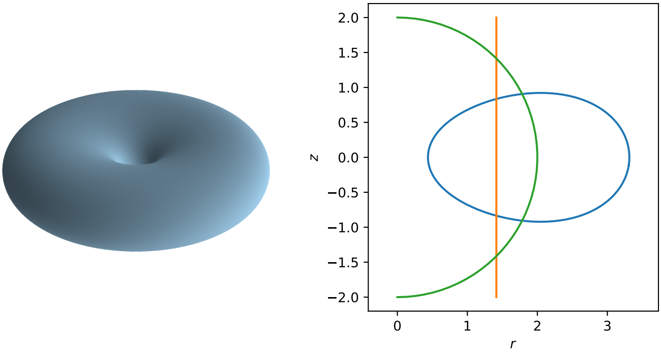

Figure: The Angenent torus (left) and its cross-section (right), with the self-shrinking sphere (green) and cylinder (orange) for comparison.

Angenent torus intuition

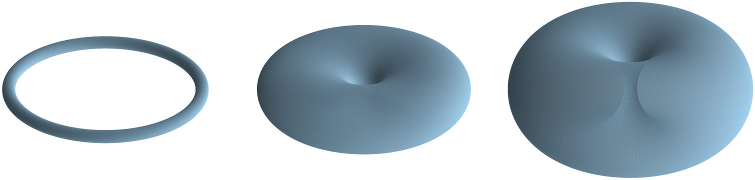

Figure: Meridian collapse (left), inner longitude collapse (right), just right (middle).

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- How unstable is it?

Critical points, stability, index

Figure: Stable critical point (left), unstable critical points (right)

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Morse index

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- The corresponding eigenvectors give unstable “downward” directions.

Toy example illustrating critical curves and stability

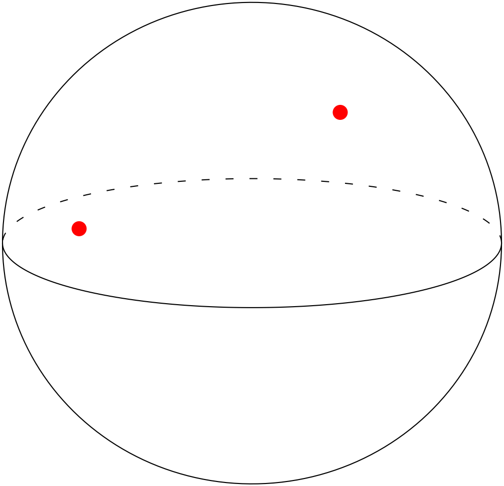

Figure: Geodesics are critical points of the length functional. Two cities can be connected with a stable geodesic and with an unstable geodesic.

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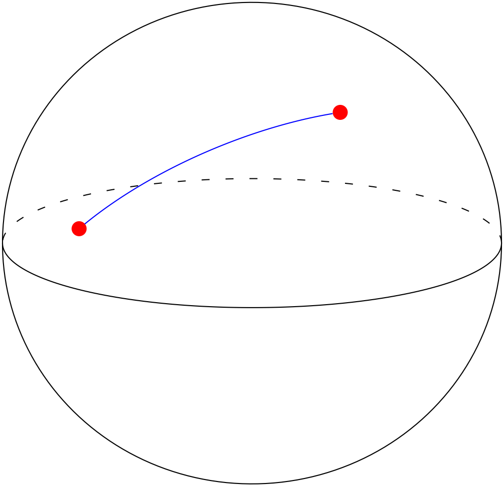

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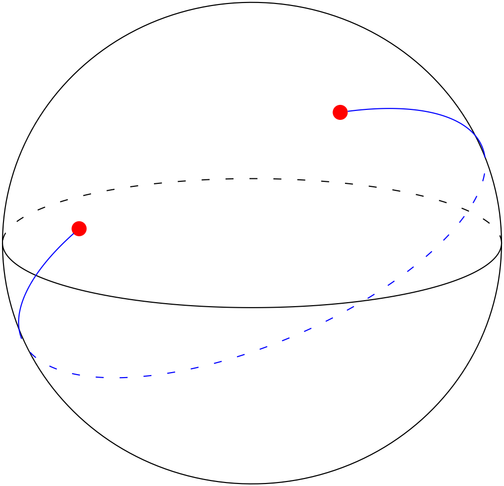

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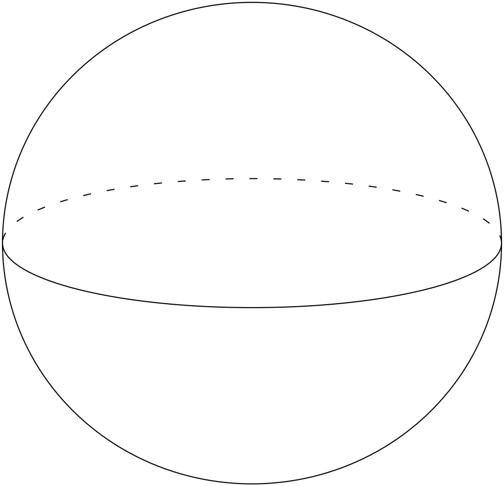

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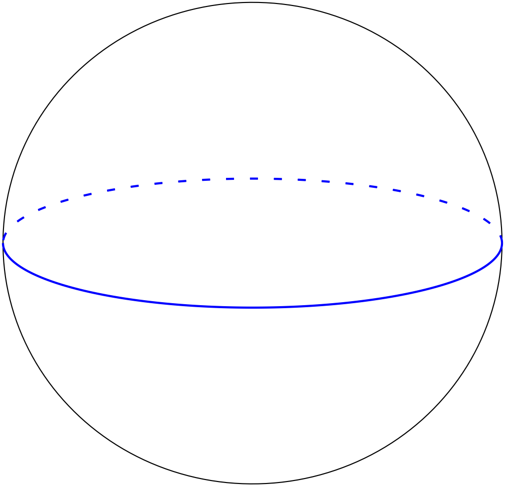

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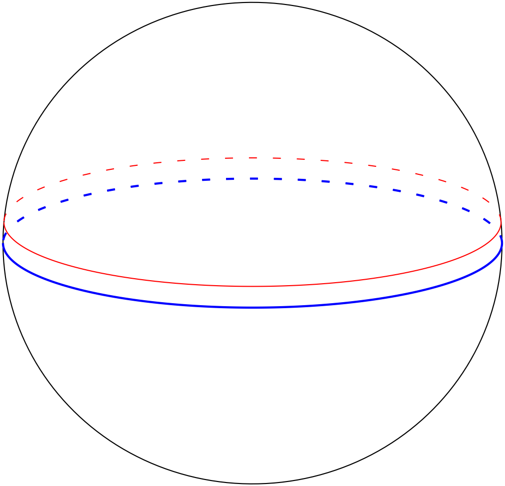

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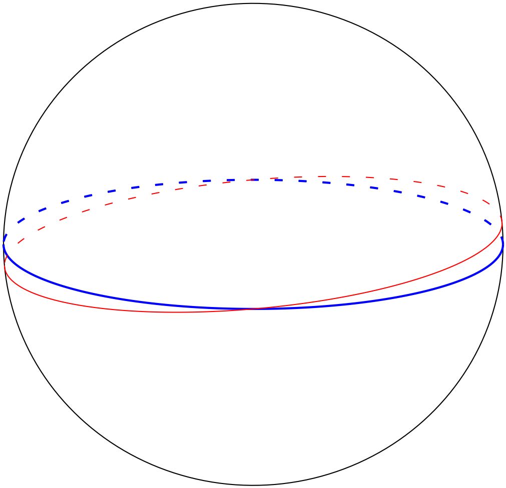

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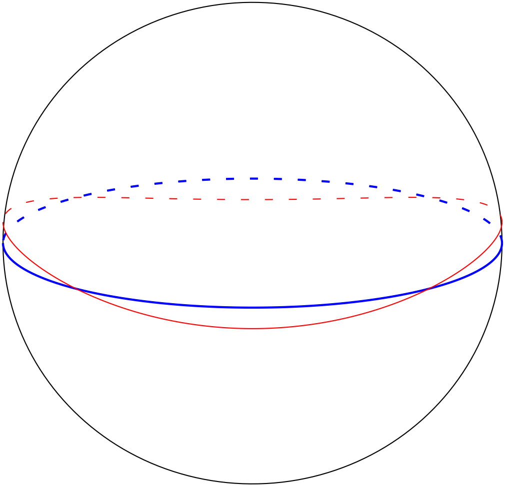

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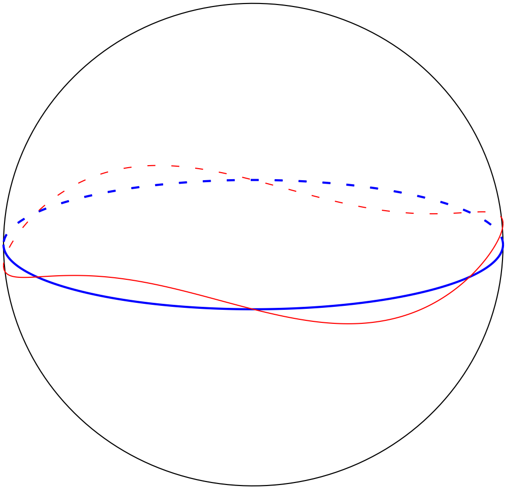

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A variational formulation for self-shrinkers

Theorem (Huisken, 1990)

A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is a self-shrinker that becomes extinct at the origin after one unit of time if and only if it is a critical point of the weighted area functional called the F-functional.

$$F(\Sigma) = (4\pi)^{-n/2} \int_\Sigma e^{-|x|^2/4} \, dArea.$$

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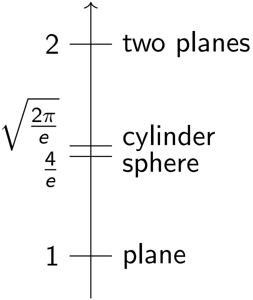
Morse index of a self-shrinker

- The index is the number of negative eigenvalues of the "Hessian".
- The corresponding eigenfunctions give variations that are unstable (decrease F).

The entropy of self-shrinkers

The critical value of the F-functional, called the entropy of the self-shrinker, is helpful in understanding what kinds of singularities can occur.



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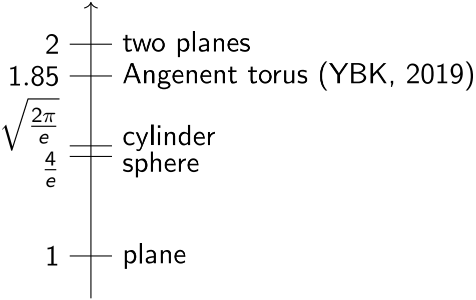

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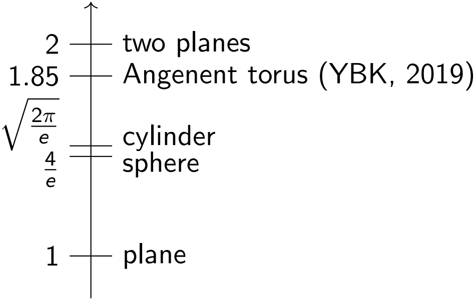

Figure: Entropies of self-shrinking surfaces

Earlier work (Drugan and Nguyen, 2018): the entropy of the Angenent torus is less than 2.

Numerical estimates of the entropy of the Angenent torus

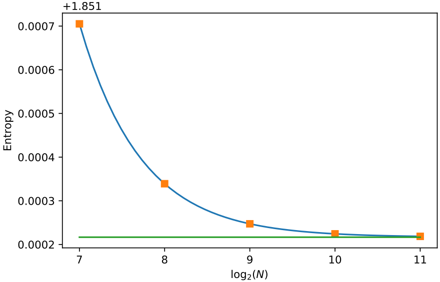

Figure: The entropy of the Angenent torus as computed using 128, 256, 512, 1024, and 2048 points. The values (orange) appear to lie on an exponential curve (blue) converging to 1.8512167 (green).

- The convergence rate suggests that the computed value is within 2×10^{-6} of the true value.
- Later work (Barrett, Deckelnick, Nürnberg, 2020) obtained the same value using different methods.

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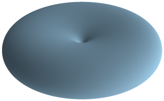
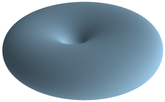
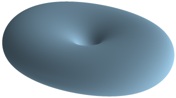
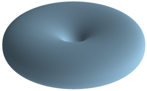
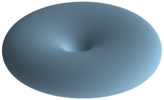
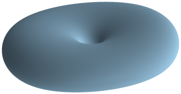
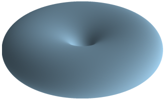
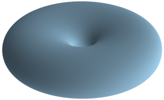
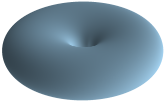
- The corresponding eigenfunctions are the unstable variations.

Index results (YBK, 2020)

$k=0$	$k=1$	$k=2$
$\lambda_0 \approx -3.740$ 	$\lambda_0 = -1$ 	$\lambda_0 \approx -0.488$ 
$\lambda_1 = -1$ 	$\lambda_0 = -1$ 	$\lambda_0 \approx -0.488$ 
$\lambda_2 = -\frac{1}{2}$ 	$\lambda_1 = -\frac{1}{2}$ 	
	$\lambda_1 = -\frac{1}{2}$ 	

Future directions

- Higher-dimensional Angenent doughnuts $S^1 \times S^{n-1} \subset \mathbb{R}^{n+1}$.
- Other self-shrinkers determined by a 1D cross-section.
- General self-shrinking surfaces (without symmetry).
- Error bounds.

Part 2

Intrinsic Geometry

Extrinsic geometry

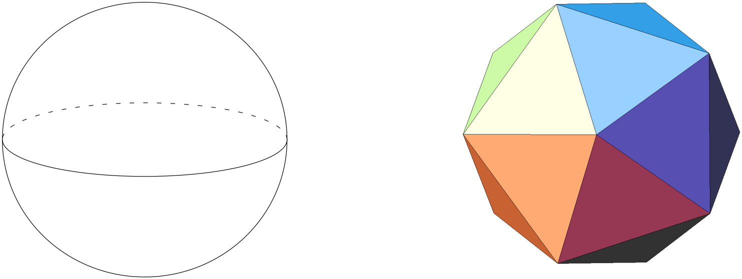
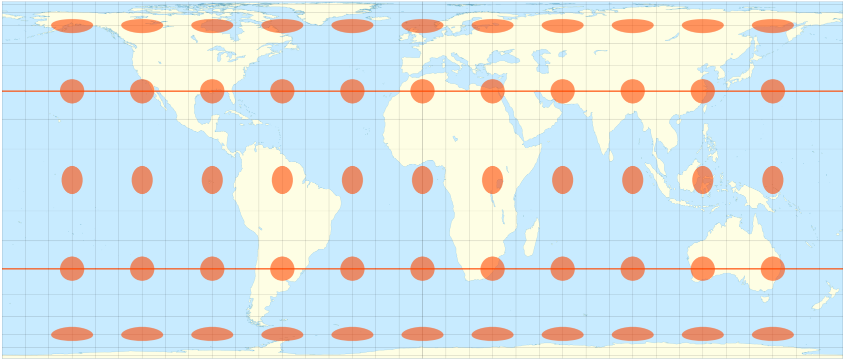


Figure: Image credit (right): Wikipedia

Intrinsic Geometry



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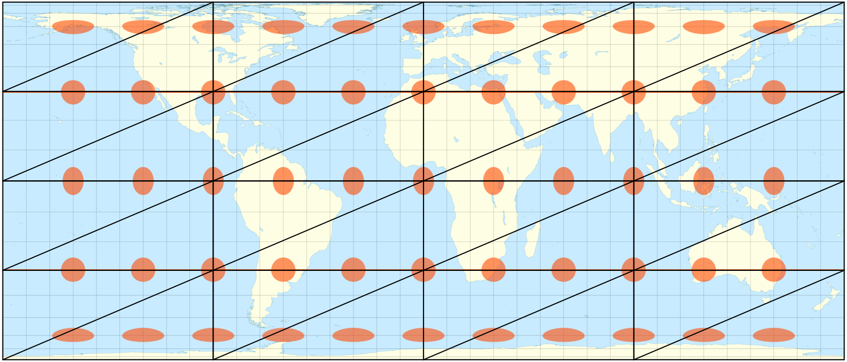

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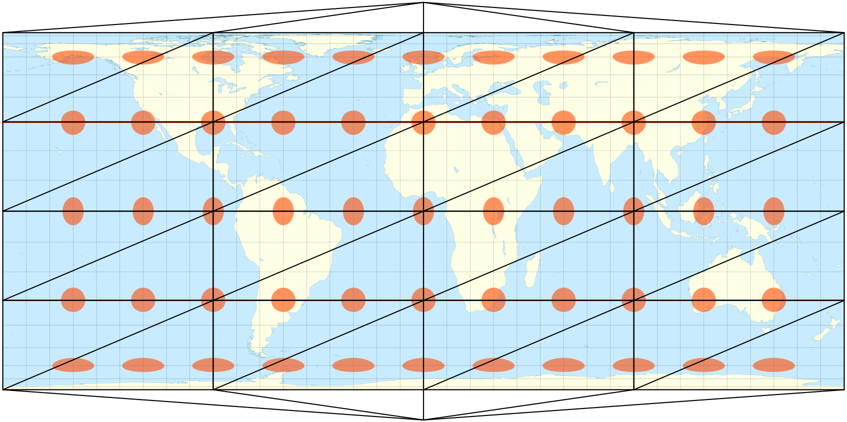

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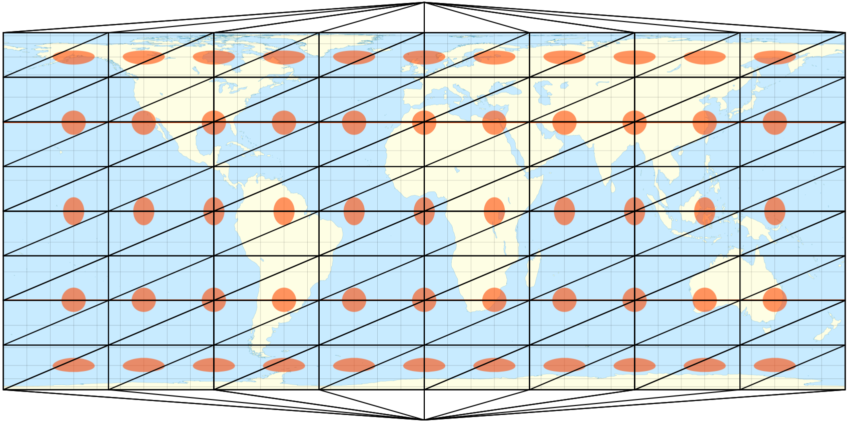

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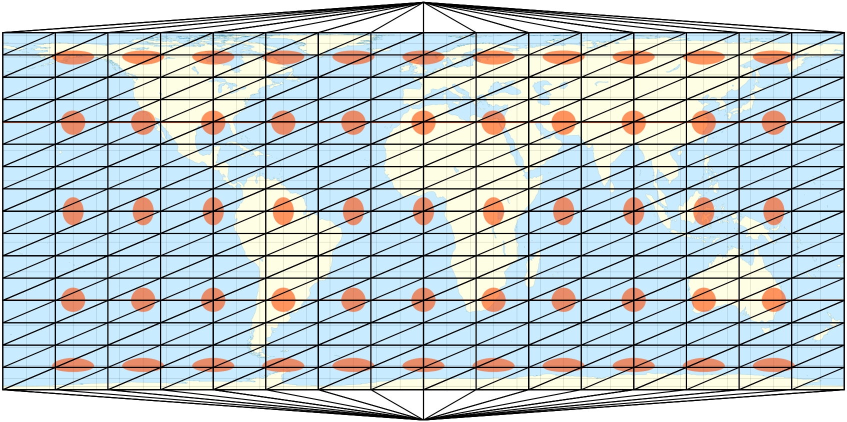

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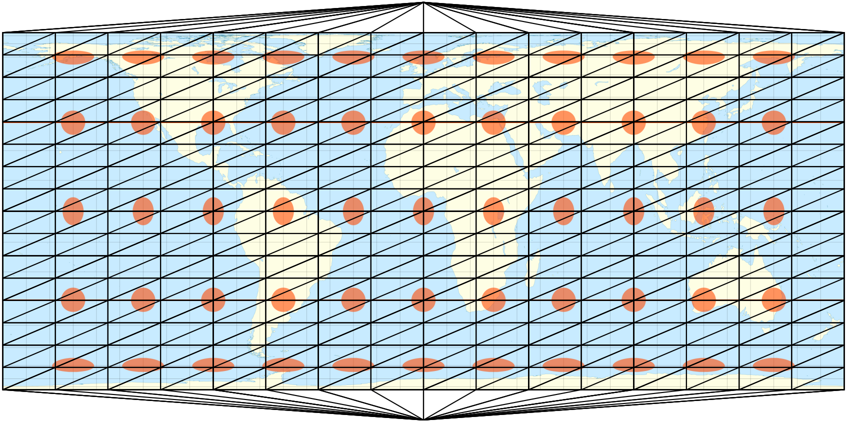

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- Regge calculus (1961): intrinsic discretization of Riemannian metrics via edge lengths.

Higher order methods: scalar functions

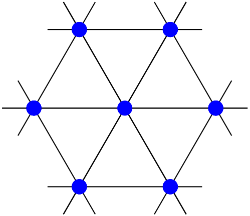

Figure: Piecewise linear on a finer mesh, or piecewise quadratics?

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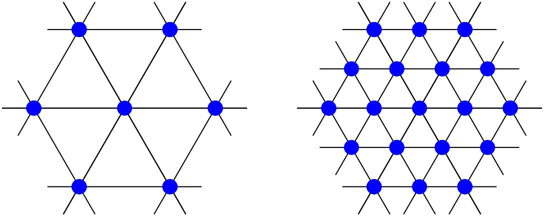

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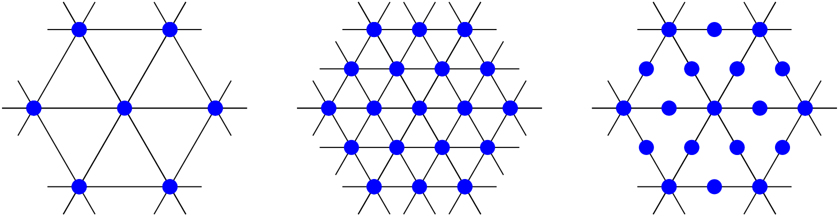

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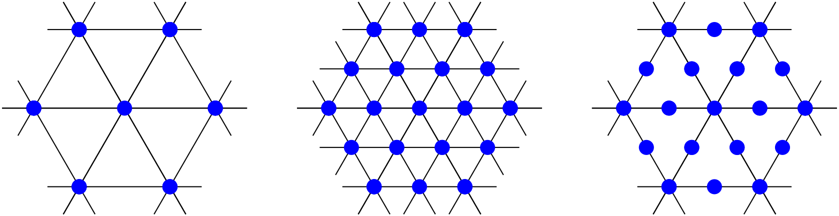

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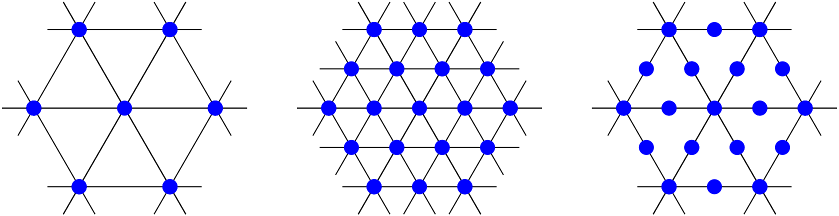

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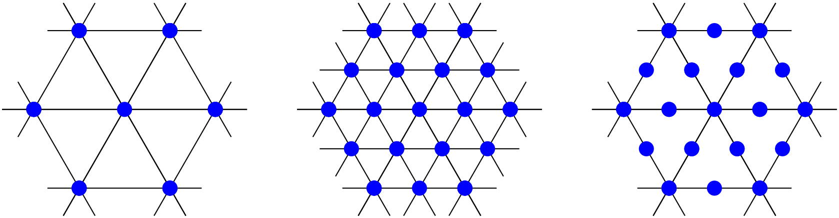

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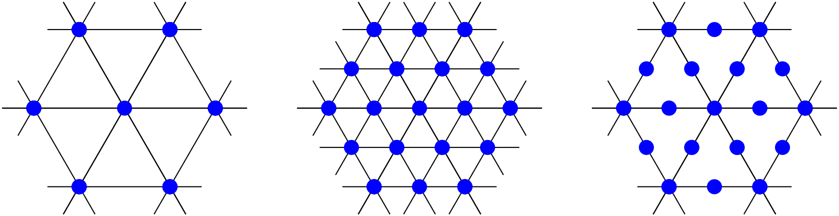

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  - e.g. mean curvature flow (Kovács, Li, Lubich, 2019).

Higher order methods for Riemannian geometry

From Regge calculus to Regge finite elements

- Piecewise constant Riemannian metrics.
  - Regge calculus (Regge, 1961).
- Piecewise polynomial Riemannian metrics.
  - Regge finite elements (Li, 2018).

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From Regge calculus to Regge finite elements

- Piecewise constant Riemannian metrics.
  - Regge calculus (Regge, 1961).
- Piecewise polynomial Riemannian metrics.
  - Regge finite elements (Li, 2018).

Goal: finite element Riemannian geometry

- Levi-Civita connection
- curvature
- Bochner Laplacian
- convergence rates

References

Yakov Berchenko-Kogan.
The entropy of the Angenent torus is approximately 1.85122.
Experimental Math., 2019.

Yakov Berchenko-Kogan.
Bounds on the index of rotationally symmetric self-shrinking tori.
Geom. Dedicata, 2021.

Yakov Berchenko-Kogan.
Numerically computing the index of mean curvature flow self-shrinkers.
Results Math., 2022.

Y. I. Berchenko-Kogan and E. Gawlik.
Finite element approximation of the Levi-Civita connection and its curvature in two dimensions.
Accepted for publication in *Foundations of Computational Math*.
https://arxiv.org/abs/2111.02512.

Thank you