Symmetry in Finite Element Exterior Calculus

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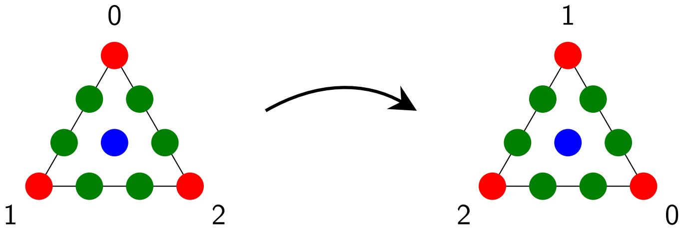
Symmetry of Scalar Elements

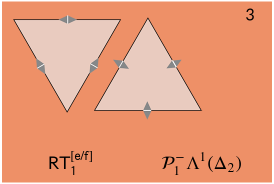


$$\mathcal{P}_3\Lambda^0(T^2) = \left\langle \lambda_0^3, \lambda_1^3, \lambda_2^3, \lambda_1^2\lambda_2, \lambda_2^2\lambda_1, \lambda_2^2\lambda_0, \lambda_0^2\lambda_2, \lambda_0^2\lambda_1, \lambda_1^2\lambda_0, \lambda_0\lambda_1\lambda_2 \right\rangle .$$

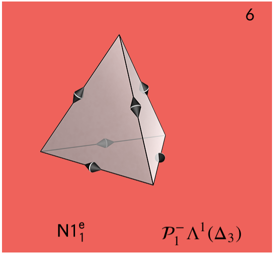
- When computing matrix of, e.g., $a(u,v) = \int_{T^2} \nabla u \cdot \nabla v$, can exploit sixfold symmetry of T^2 to compute fewer entries.

$$\begin{aligned} a\left(\lambda_0^3, \lambda_1^2\lambda_2\right) &= a\left(\lambda_0^3, \lambda_2^2\lambda_1\right) \\ = a\left(\lambda_1^3, \lambda_2^2\lambda_0\right) &= a\left(\lambda_1^3, \lambda_0^2\lambda_2\right) \\ = a\left(\lambda_2^3, \lambda_0^2\lambda_1\right) &= a\left(\lambda_2^3, \lambda_1^2\lambda_0\right) \end{aligned}$$

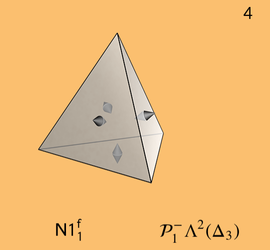
Symmetry of Vector Elements
Whitney Elements



$$\left\langle \lambda_1\, d\lambda_2 - \lambda_2\, d\lambda_1,\ \lambda_2\, d\lambda_0 - \lambda_0\, d\lambda_2,\ \lambda_0\, d\lambda_1 - \lambda_1\, d\lambda_0 \right\rangle.$$

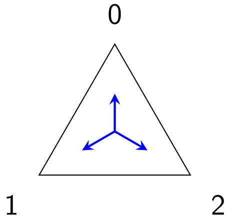


$$\left\langle \lambda_1\, d\lambda_2 - \lambda_2\, d\lambda_1,\ \lambda_2\, d\lambda_0 - \lambda_0\, d\lambda_2,\ \lambda_0\, d\lambda_1 - \lambda_1\, d\lambda_0,\ \lambda_0\, d\lambda_3 - \lambda_3\, d\lambda_0,\ \lambda_1\, d\lambda_3 - \lambda_3\, d\lambda_1,\ \lambda_2\, d\lambda_3 - \lambda_3\, d\lambda_2 \right\rangle.$$



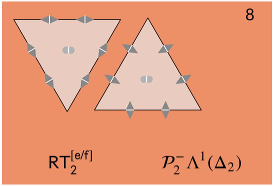
$$\langle\ \lambda_1\, d\lambda_2 \wedge d\lambda_3 + \lambda_2\, d\lambda_3 \wedge d\lambda_1 + \lambda_3\, d\lambda_1 \wedge d\lambda_2,\ \ldots,\ \lambda_0\, d\lambda_1 \wedge d\lambda_2 + \lambda_1\, d\lambda_2 \wedge d\lambda_0 + \lambda_2\, d\lambda_0 \wedge d\lambda_1 \rangle$$

Geometric symmetry $\Rightarrow$ basis symmetry (up to sign).

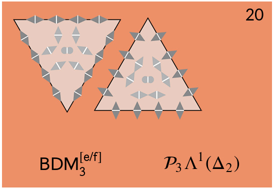


$$\mathcal{P}_0\Lambda^1(T^2) = \langle d\lambda_0, d\lambda_1, d\lambda_2\rangle,$$

$$d\lambda_0 + d\lambda_1 + d\lambda_2 = 0$$



$$\begin{aligned}\langle\, &\lambda_1^2\, d\lambda_2 - \lambda_1\lambda_2\, d\lambda_1,\\ &\lambda_2^2\, d\lambda_1 - \lambda_1\lambda_2\, d\lambda_2,\\ &\ldots,\\ &\lambda_0^2\, d\lambda_1 - \lambda_0\lambda_1\, d\lambda_0,\\ &\lambda_1^2\, d\lambda_0 - \lambda_0\lambda_1\, d\lambda_1,\\ &\lambda_0\lambda_1\, d\lambda_2 - \lambda_0\lambda_2\, d\lambda_1,\\ &\lambda_1\lambda_2\, d\lambda_0 - \lambda_0\lambda_1\, d\lambda_2,\\ &\lambda_0\lambda_2\, d\lambda_1 - \lambda_1\lambda_2\, d\lambda_0\,\rangle.\end{aligned}$$



$$\begin{aligned}\langle\, &\ldots,\\ &\ldots,\\ &\lambda_0\lambda_1\lambda_2\, d\lambda_0,\\ &\lambda_0\lambda_1\lambda_2\, d\lambda_1,\\ &\lambda_0\lambda_1\lambda_2\, d\lambda_2\,\rangle.\end{aligned}$$

Results

Theorem (if: Licht, 2019; only if: YBK, 2021)

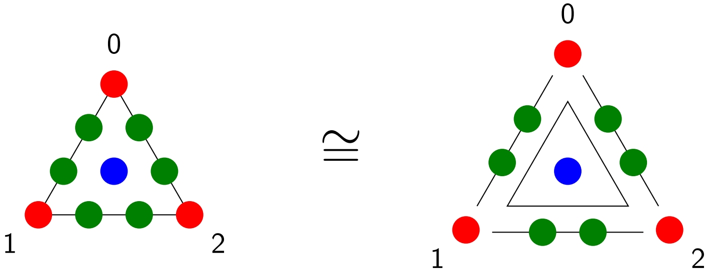
The following spaces have symmetry-invariant bases up to sign if and only if the corresponding condition holds.

$$\begin{array}{lcl} \mathcal{P}_r\Lambda^1(T^2) & \textit{if and only if} & r \notin 3\mathbb{N}_0, \\ \mathcal{P}_r^-\Lambda^1(T^2) & \textit{if and only if} & r \notin 3\mathbb{N}_0 + 2. \end{array}$$

Theorem (YBK, 2021)

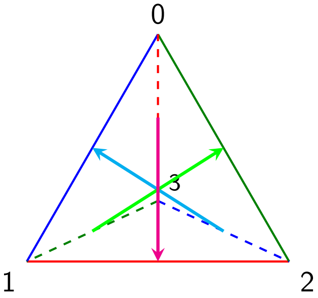
The following spaces have symmetry-invariant bases up to sign if and only if the corresponding condition holds.

$$\begin{array}{lcl} \mathcal{P}_r\Lambda^1(T^3) & \textit{always}, & \\ \mathcal{P}_r^-\Lambda^1(T^3) & \textit{if and only if} & r \notin 3\mathbb{N}_0 + 2, \\ \mathcal{P}_r\Lambda^2(T^3) & \textit{always}, & \\ \mathcal{P}_r^-\Lambda^2(T^3) & \textit{always}. & \end{array}$$



$$
\begin{aligned}
\mathcal{P}_3\Lambda^0(T^2) &\cong 3\mathring{\mathcal{P}}_3\Lambda^0(T^0) \oplus 3\mathring{\mathcal{P}}_3\Lambda^0(T^1) \oplus \mathring{\mathcal{P}}_3\Lambda^0(T^2) \\
&\cong 3\mathcal{P}_2\Lambda^0(T^0) \oplus 3\mathcal{P}_1\Lambda^1(T^1) \oplus \mathcal{P}_0\Lambda^2(T^2)
\end{aligned}
$$

$$
\begin{aligned}
&\langle \lambda_0^3 \rangle \oplus \langle \lambda_1^3 \rangle \oplus \langle \lambda_2^3 \rangle \\
&\quad \oplus \langle \lambda_1^2\lambda_2, \lambda_2^2\lambda_1 \rangle \oplus \langle \lambda_2^2\lambda_0, \lambda_0^2\lambda_2 \rangle \oplus \langle \lambda_0^2\lambda_1, \lambda_1^2\lambda_0 \rangle \oplus \langle \lambda_0\lambda_1\lambda_2 \rangle \\
\cong\ &\langle \lambda_0^2 \rangle \oplus \langle \lambda_1^2 \rangle \oplus \langle \lambda_2^2 \rangle \\
&\quad \oplus \langle \lambda_1\, ds, \lambda_2\, ds \rangle \oplus \langle \lambda_0\, ds, \lambda_2\, ds \rangle \oplus \langle \lambda_0\, ds, \lambda_1\, ds \rangle \oplus \langle 1\, dA \rangle
\end{aligned}
$$



$$\mathcal{P}_0\Lambda^1(T^3)$$
$$= \langle d\lambda_0 + d\lambda_1 - d\lambda_2 - d\lambda_3,$$
$$d\lambda_0 + d\lambda_2 - d\lambda_1 - d\lambda_3,$$
$$d\lambda_1 + d\lambda_2 - d\lambda_0 - d\lambda_3\rangle$$
$$=: \langle \alpha, \beta, \gamma\rangle.$$

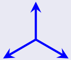
$$\mathcal{P}_2\Lambda^1(T^3)$$
$$= \mathcal{P}_2\Lambda^0(T^3) \otimes \mathcal{P}_0\Lambda^1(T^3)$$
$$= \langle \lambda_0^2\alpha, \lambda_0^2\beta, \lambda_0^2\gamma,$$
$$\lambda_1^2\alpha, \lambda_1^2\beta, \lambda_1^2\gamma,$$
$$\lambda_2^2\alpha, \lambda_2^2\beta, \lambda_2^2\gamma,$$
$$\lambda_3^2\alpha, \lambda_3^2\beta, \lambda_3^2\gamma,$$
$$\lambda_0\lambda_1\alpha, \lambda_0\lambda_1\beta, \lambda_0\lambda_1\gamma,$$
$$\lambda_0\lambda_2\alpha, \lambda_0\lambda_2\beta, \lambda_0\lambda_2\gamma,$$
$$\lambda_0\lambda_3\alpha, \lambda_0\lambda_3\beta, \lambda_0\lambda_3\gamma,$$
$$\lambda_1\lambda_2\alpha, \lambda_1\lambda_2\beta, \lambda_1\lambda_2\gamma,$$
$$\lambda_1\lambda_3\alpha, \lambda_1\lambda_3\beta, \lambda_1\lambda_3\gamma,$$
$$\lambda_2\lambda_3\alpha, \lambda_2\lambda_3\beta, \lambda_2\lambda_3\gamma\rangle.$$

Representations of $\mathbb{Z}/3$

- The 1D representation $\mathbf{1}$ where $\mathbb{Z}/3$ acts trivially.
- The 2D representation $\mathbf{2}$ where $\mathbb{Z}/3$ acts by 120° rotations.
- The 3D representation $\mathbf{3}$ where $\mathbb{Z}/3$ acts by permuting the coordinates.
    - $\mathbf{3} \cong \mathbf{1} \oplus \mathbf{2}$ because $\langle(1,1,1)\rangle$ is an invariant subspace.

Invariant bases

$\mathbf{1}$ and $\mathbf{3}$ have symmetry-invariant bases, but $\mathbf{2}$ does not.



Proposition

A representation $V \cong m\mathbf{1} \oplus n\mathbf{2}$ has a $\mathbb{Z}/3$-invariant basis up to sign if and only if $m \geq n$.

Thank You and References

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