Duality in Finite Element Exterior Calculus and the Hodge Star Operator on the Sphere

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The finite element method

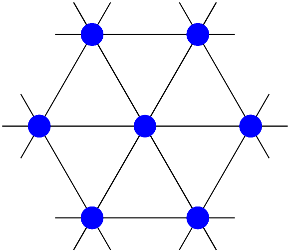

Figure: Degrees of freedom (blue) of piecewise linear functions (left) and piecewise quadratic functions (right).

Geometric decomposition

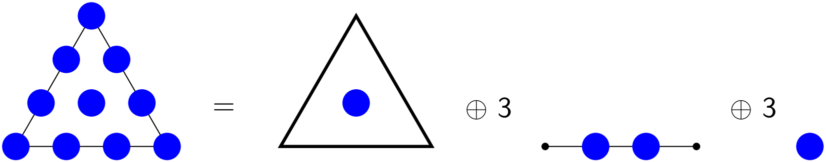

Figure: $\mathcal{P}_3(T^2) = \mathring{\mathcal{P}}_3(T^2) \oplus 3\mathring{\mathcal{P}}_3(T^1) \oplus 3\mathring{\mathcal{P}}_3(T^0)$

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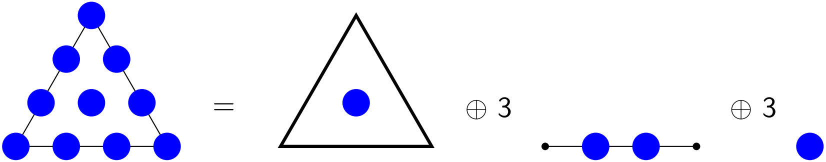

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$$(\mathcal{P}_3(T^2))^* \;\cong\; (\mathring{\mathcal{P}}_3(T^2))^* \;\oplus\; 3(\mathring{\mathcal{P}}_3(T^1))^* \;\oplus\; 3(\mathring{\mathcal{P}}_3(T^0))^*$$

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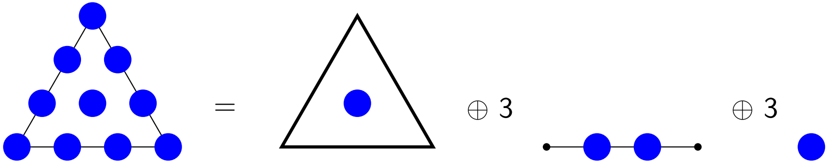

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&\cong \mathcal{P}_0(T^2) \oplus 3\mathcal{P}_1(T^1) \oplus 3\mathcal{P}_2(T^0)
\end{aligned}$$

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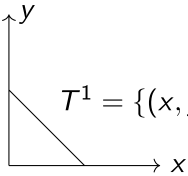
Problem

Construct a bijection $\mathcal{P} \to \mathcal{Q}$ so that for nonzero $p \mapsto q$ we have

- q only depends on p pointwise, and
- $\int_{T^n} pq > 0$.

Explicit pointwise duality



$$T^1 = \{(x, y) \mid x + y = 1\}$$

Figure: Barycentric coordinates

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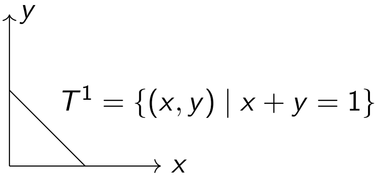

Figure: Barycentric coordinates

Example (Duality between $\mathcal{P}_1(T^1)$ and $\mathring{\mathcal{P}}_3(T^1)$)

For $p \in \mathcal{P}_1(T^1)$, set $q = (xy)p$. Likewise, given q, set $p = \frac{q}{xy}$.

$\mathcal{P}_1(T^1)$	$\mathring{\mathcal{P}}_3(T^1)$	$\int_{T^1} pq$
x	x^2y	$\int_{T^1}(xy)x^2$
y	xy^2	$\int_{T^1}(xy)y^2$

Finite element exterior calculus

Spaces $\mathcal{P}_r\Lambda^k(T^n)$ and $\mathcal{P}_r^-\Lambda^k(T^n)$ of k-forms on T^n with polynomial coefficients of degree at most r.

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- scalar fields
  - Lagrange
  - Discontinuous Galerkin

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  - Brezzi–Douglas–Marini elements
  - Raviart–Thomas elements
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$\mathcal{P}_r\Lambda^1(T^3)$ and $\mathcal{P}_r^-\Lambda^1(T^3)$ are Nédélec $H(\mathrm{curl})$ elements of the 2nd and 1st kinds, respectively.

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- See Arnold, Falk, Winther, 2006.

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Theorem (Arnold, Falk, and Winther)

With respect to the integration pairing

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Construct an explicit bijection between these spaces so that for nonzero $a \mapsto b$ we have

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A motivating example

Let Ω be an 3-dimensional domain.

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- .
$$\int_\Omega a \wedge b = \int_\Omega (a_x^2 + a_y^2 + a_z^2)\, d\mathrm{vol} = \int_\Omega \|a\|^2\, d\mathrm{vol} > 0,$$

The simplex and the sphere

- T^2 consists of points in the first orthant with $x + y + z = 1$.
- Via the change of coordinates

$$x = u^2, \qquad y = v^2, \qquad z = w^2,$$

we obtain the unit sphere $u^2 + v^2 + w^2 = 1$.

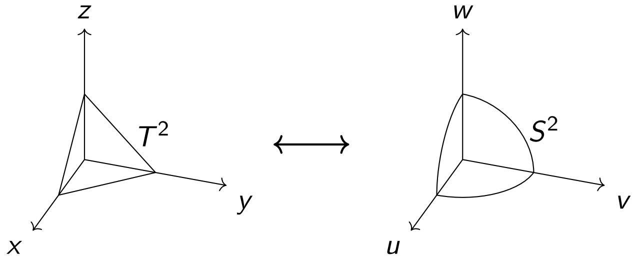

Figure: Change of coordinates

Explicit pointwise duality for finite element exterior calculus

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Theorem (YBK)

- *b depends on a pointwise.*
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5. $b = xy^2\,dz - y^2z\,dx \in \mathring{\mathcal{P}}_3^-\Lambda^1(T^2)$.

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5. $b = xy\,dz \in \mathring{\mathcal{P}}_2\Lambda^1(T^2)$.

Integration via u-substitution

$$\int_{T^2} a \wedge b = \int_{S^2_{>0}} \alpha \wedge \beta = \int_{S^2_{>0}} uvw(\alpha \wedge *_{S^2}\alpha)$$

$$= \int_{S^2_{>0}} uvw\,\|\alpha\|^2 \; d\mathrm{Area} > 0$$

Correspondence between forms on T^n and forms on S^n

Change of coordinates

$$x = u^2, \qquad y = v^2, \qquad z = w^2,$$
$$dx = 2u\,du, \qquad dy = 2v\,dv, \qquad dz = 2w\,dw.$$

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Theorem (YBK)

The change of coordinates induces a bijection between $\mathcal{P}_r\Lambda^k(T^n)$ and $\mathcal{P}_{2r+k}\Lambda^k_e(S^n)$.

Even and odd forms on the sphere

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$\alpha \in \Lambda^k(S^n)$ is odd if it changes sign under each coordinate reflection. Let $\Lambda^k_o(S^n)$ denote the space of such forms.

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- u_N is odd. □

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(optional slide) Alternate characterizations of $\mathcal{P}_r^-\Lambda^k(T^n)$

Definition

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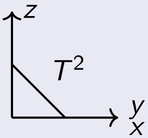
Definition

- Let X be the radial vector field

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$



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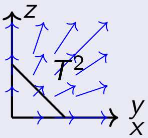

(optional slide) Alternate characterizations of $\mathcal{P}_r^- \Lambda^k(T^n)$

Definition

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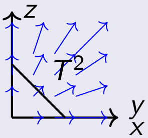
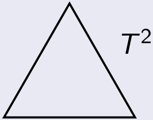


- Let X_T be the projection of the radial vector field to T^n.

$$X_T = \left(x - \tfrac{1}{3}\right)\frac{\partial}{\partial x} + \left(y - \tfrac{1}{3}\right)\frac{\partial}{\partial y} + \left(z - \tfrac{1}{3}\right)\frac{\partial}{\partial z}.$$



(optional slide) Alternate characterizations of $\mathcal{P}_r^-\Lambda^k(T^n)$

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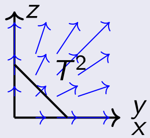

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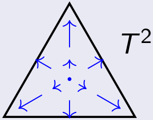

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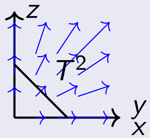

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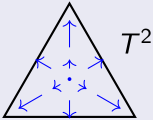

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(optional slide) Alternate characterizations of $\mathcal{P}_r^-\Lambda^k(T^n)$

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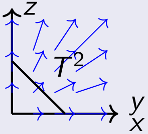

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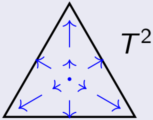

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Definition (Arnold, Falk, and Winther)

$$\mathcal{P}_r^-\Lambda^k(T^n) := \mathcal{P}_{r-1}\Lambda^k(T^n) + i_{X_T}\mathcal{P}_{r-1}\Lambda^{k+1}(T^n).$$

(optional slide) Alternate characterizations of $\mathcal{P}_r^-\Lambda^k(T^n)$

$\mathcal{P}_r\Lambda^k(T^n)$ is the restriction of $\mathcal{P}_r\Lambda^k(\mathbb{R}^{n+1})$ to T^n. Likewise...

(optional slide) Alternate characterizations of $\mathcal{P}_r^-\Lambda^k(T^n)$

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Theorem (YBK)

The two definitions are equivalent.

Thank you